Exercises

Taylor Approximation – Solutions

Exercise 1.

(a) $f(x) = \ln(x), \qquad n = 2, \qquad x_0 = 1.$

k	$f^{(k)}(x)$	$f^{(k)}(x_0)$
0	$\ln(x)$	0
1	$\frac{1}{x}$	1
2	$-\frac{1}{x^2}$	-1

thus

$$\begin{aligned} T_{2,1}(x) &= \frac{0}{0!}(x-1)^0 + \frac{1}{1!}(x-1)^1 + \frac{-1}{2!}(x-1)^2 \\ &= 0 + (x-1) - \frac{1}{2}(x-1)^2 \end{aligned}$$

(b) $f(x) = \sin(x), \qquad n = 1, \ldots, 4, \qquad x_0 = 0.$

k	$f^{(k)}(x)$	$f^{(k)}(x_0)$
0	$\sin(x)$	0
1	$\cos(x)$	1
2	$-\sin(x)$	0
3	$-\cos(x)$	-1
4	$\sin(x)$	0

$$\begin{aligned} T_{1,0}(x) &= \frac{0}{0!}(x-0)^0 + \frac{1}{1!}(x-0)^1 = x \\ T_{2,0}(x) &= T_{1,0}(x) + \frac{0}{2!}(x-0)^2 = x \\ T_{3,0}(x) &= T_{2,0}(x) + \frac{-1}{3!}(x-0)^3 = x - \frac{1}{3}x^3 \\ T_{4,0}(x) &= T_{3,0}(x) + \frac{0}{4!}(x-0)^4 = x - \frac{1}{3}x^3 \end{aligned}$$

(c) $f(x) = 4x^4 - 3x^3 + x^2 - 10x + 42, \qquad n = 2, \ldots, 5, \qquad x_0 = 0$

k	$f^{(k)}(x)$	$f^{(k)}(x_0)$
0	$4x^4 - 3x^3 + x^2 - 10x + 42$	42
1	$16x^3 - 9x^2 + 2x - 10$	-10
2	$48x^2 - 18x + 2$	2
3	$96x - 18$	-18
4	96	96
5	0	0

$$
\begin{aligned}
T_{2,0}(x) &= \frac{42}{0!}(x-0)^0 + \frac{-10}{1!}(x-0)^1 + \frac{2}{2!}(x-0)^2 = 42 - 10x + x^2 \\
T_{3,0}(x) &= T_{2,0}(x) + \frac{-18}{3!}(x-0)^3 = 42 - 10x + x^2 + 3x^3 \\
T_{4,0}(x) &= T_{3,0}(x) + \frac{96}{4!}(x-0)^4 = 42 - 10x + x^2 + 3x^3 + 4x^4 \\
T_{5,0}(x) &= T_{4,0}(x) + \frac{0}{5!}(x-0)^5 = 42 - 10x + x^2 + 3x^3 + 4x^4 + 0
\end{aligned}
$$

(d) $f(x) = e^x, \qquad n = 2, \qquad x_0 = 0$

k	$f^{(k)}(x)$	$f^{(k)}(x_0)$
0	xe^x	0
1	$e^x(1+x)$	1
2	$e^x(2+x)$	2

$$
\begin{aligned}
T_{2,1}(x) &= \frac{0}{0!}(x-0)^0 + \frac{1}{1!}(x-0)^1 + \frac{2}{2!}(x-0)^2 \\
&= 0 + x + x^2
\end{aligned}
$$

Exercise 2.

(a) (i) $f(x) = \sin(x), \qquad n = 3, \qquad x_0 = \pi.$

k	$f^{(k)}(x)$	$f^{(k)}(x_0)$
0	$\cos(x)$	1
1	$-\sin(x)$	0
2	$-\cos(x)$	-1
3	$\sin(x)$	0
4	$\cos(x)$	-

$$
T_{3,\pi}(x) = 1 + 0 - \frac{1}{2}(x-\pi)^2 + 0 = 1 + \frac{1}{2}(x-\pi)^2
$$

(ii) The remainder of $T_{3,\pi}$ is

$$R_{3,\pi}(x,\xi) = \frac{\cos(\xi)}{4!}(x-\pi)^4$$

and thus

$$\begin{aligned}
|R_{3,\pi}(x,\xi)| &= \frac{|\cos(\xi)|}{24}(x-\pi)^4 \\
&\le \frac{1}{24}(x-\pi)^4 \\
&\le \frac{1}{24}(2\pi-\pi)^4 \\
&= \frac{\pi^4}{24} \approx 4.05
\end{aligned}$$

The maximal error $|f(x) - T_{3,\pi}(x)|$ is $\frac{\pi^4}{24} \approx 4.05$ on the interval $[0, 2\pi]$.

(b) (i) $f(x) = \frac{1}{1-x}$, $\quad n = 2$, $\quad x_0 = 4$.

k	$f^{(k)}(x)$	$f^{(k)}(x_0)$
0	$\frac{1}{1-x}$	$-\frac{1}{3}$
1	$\frac{1}{(1-x)^2}$	$\frac{1}{9}$
2	$\frac{2}{(1-x)^3}$	$-\frac{2}{27}$
3	$\frac{6}{(1-x)^4}$	-

$$T_{2,4}(x) = -\frac{1}{3} + \frac{1}{9}(x-4) - \frac{1}{27}(x-4)^2$$

(ii) The remainder of $T_{2,4}$ is

$$R_{2,4}(x,\xi) = \frac{1}{6}\frac{6}{(1-\xi)^4}(x-4)^3$$

and thus

$$\begin{aligned}
|R_{2,4}(x,\xi)| &= \frac{1}{|1-\xi|^4}|x-4|^3 \\
&\le \max_{\xi\in[3,5]} \frac{1}{|1-\xi|^4} \max_{x\in[3,5]} |x-4|^3 \\
&= \frac{1}{|1-3|^4} \cdot |5-4|^3 \\
&= \frac{1}{16} \cdot 1 = \frac{1}{16}
\end{aligned}$$

where we used that the remainder is less or equal than the term if we maximize with respect to ξ and x separately. The term $\frac{1}{|1-\xi|^4}$ is maximal for the number ξ closest to 1, i.e. in this case the smallest possible ξ. This is $\xi = 3$. The term $|x-4|^4$ is maximal for the number with the greatest distance to 4 — in this case for $x = 5$ (or $x = 3$).

Exercise 3.

1. $f(x) = \sqrt{1+x}, \qquad n = 1, \qquad x_0 = 0.$

k	$f^{(k)}(x)$	$f^{(k)}(x_0)$
0	$\sqrt{1+x}$	1
1	$\frac{1}{2\sqrt{1+x}}$	$\frac{1}{2}$
2	$-\frac{1}{4(1+x)\sqrt{1+x}}$	-

$$T_{1,0}(x) = 1 + \frac{1}{2}x^2$$
$$R_{1,0}(x,\xi) = -\frac{1}{4(1+\xi)\sqrt{1+\xi}}x^2$$

for some $\xi \in [0, x]$ and $x \geq 0$.

2. By the Taylor formula we have

$$\begin{aligned}\sqrt{1+x} = f(x) &= T_{1,0}(x) + R_{1,0}(x,\xi)\\ &= 1 + \frac{1}{2}x^2 - \underbrace{\frac{1}{4(1+\xi)\sqrt{1+\xi}}x^2}_{\geq 0}\\ &\leq 1 + \frac{1}{2}x^2\end{aligned}$$

where we used that $\xi \geq x_0 = 0$ and $x^2 \geq 0$.